Radiation of acoustic waves

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The following sections are based on different editions of Kinsler (2000).

1 Pulsating sphere

We will first derive an exact solution of the sound field of a pulsating sphere. The exact solution will then be taken to develop an approximate solution for the far field of a sphere.

1.1 Exact solution

We take as a model the pulsating sphere:

$$u(t) = U_0 e^{j\omega t} \tag{1}$$

where $u(t)$ is the velocity on the sphere, U_0 the amplitude and ω the frequency. We assume that the pressure waves only depend on r. We would like to know the pressure $p(r,t)$ at any point r in space.

$$p(r,t) = \frac{A}{r} e^{j(\omega t - kr)} \tag{2}$$

It'll be now the task to determine the value of the constant A. With the help of the impedance of spherical waves:

$$z(r) = p/u = \rho_0 c \cos\Theta e^{j\Theta}, \qquad \frac{1}{kr} = \tan\Theta \tag{3}$$

we can directly calculate the pressure on the surface of the sphere:

$$p(a) = U_0 \rho_0 c \cos\Theta_a e^{j(\omega t + \Theta_a)} \tag{4}$$

By equating Eq. 2 and Eq. 4 we get for A:

$$A = \rho_0 c U_0 a \cos\Theta_a e^{j(ka+\Theta_a)} \tag{5}$$

and we can write the pressure field of a pulsating sphere as:

$$p(r,t) = \rho_0 c U_0 \frac{a}{r} \cos\Theta_a e^{j(\omega t - k(r-a)+\Theta_a)}, \qquad \frac{1}{\tan\Theta} = kr \tag{6}$$

1.2 Approximate solution for long wavelengths

In this section we'll approximate Eq. 6 for the long wavelength limit:

$$ka << 1, \qquad \lambda \to \infty \tag{7}$$

The term $\cos\Theta_a$ can be expressed in terms of ka by using Eq. 3:

$$\cos\Theta_a = \frac{ak}{\sqrt{(ak)^2+1}} \approx ak \tag{8}$$

if one accepts that $(ak)^2 << ak$. The value of Θ converges $\to \pi/2$ as $1/ka = \tan\Theta \to \infty$. Therefore the exponential $e^{j\Theta_a}$ in Eq. 6 becomes j, just a multiplicative factor. Thus, we can eliminate all terms involving Θ and the pressure field of a pulsating sphere for the long wavelength limit is:

$$p(r,t) = \frac{j\rho_0 c U_0 a^2 k}{r} e^{j(\omega t - kr)} \tag{9}$$

2 Source strength

Eq. 9 contains terms which belong to the source and other terms which belong to the medium. Obviously, a^2 and U_0 are properties of the source. They determine the strength of the source. Thus, it makes sense to introduce the source strength as:

$$Q = \int_S \vec{u} \cdot d\vec{S} \tag{10}$$

where $\vec{u}$ is the velocity vector at the surface element $d\vec{S}$. With the source strength Q the pressure can be rewritten as:

$$p(r,t) = Q \frac{j\rho_0 ck}{4\pi r} e^{j(\omega t - kr)} \tag{11}$$

Note, that we are dividing by 4π because the surface of a sphere is $4\pi a^2$ but we only have a^2 in Eq. 9. Also note that the factor 4π has to be adjusted if we are only integrating over parts of the surface. For example, if the sphere is fitted into a wall we are usually ignoring the sound which radiates to the other side of the wall. In such a case we have to use 2π.

The source strength can also be written in differential form. This means that we are considering only a small surface element $d\vec{S}$ which is vibrating at speed $\vec{u}$. We ask how much pressure $dp(r,t)$ arrives at r from the the surface $d\vec{S}$.

$$dp(r,t) = \frac{j\rho_0 ck}{4\pi r}(\vec{u} \cdot d\vec{S})e^{j(\omega t - kr)} \tag{12}$$

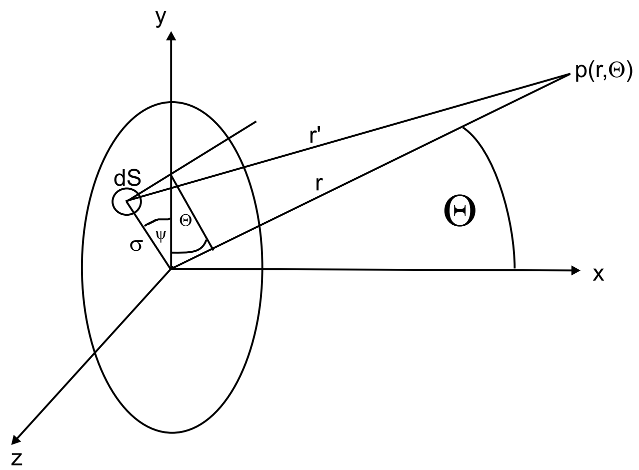

Figure 1: Radiation from a piston. Note that the angle ψ lies in the y/z-plane and the angle Θ lies in the x/y plane. To get the difference between r and r' the radius σ is first projected on the y axis and then projected on the vector r.

3 Radiation from a piston

Fig. 1 shows the geometry to calculate the radiation from a piston. We start with a small bit of surface dS on the piston which is described by the two coordinates σ and ψ. This surface is observed from the point $p(r,\Theta)$. The radius of the piston is a.

The pressure $dp(r,\Theta)$ generated by dS at σ, ψ is given by the differential source strength. However, in contrast to Eq. 12 we are looking only at one side of the piston. Thus, the normalised surface is 2π and not 4π:

$$dp(r', t) = \frac{j\rho_0 ck}{2\pi r'}(\vec{u} \cdot d\vec{S})e^{j(\omega t - kr')} \tag{13}$$

where r' is the distance between the surface dS and the point $p(r, \Theta)$. The distance r' is given by:

$$r' = r - \sigma \cos\psi \sin\Theta \tag{14}$$

where this equation reflects basically two projections: First the projection of dS on the y-axis ($\cos\psi$) and then this projection projected on the r-vector of $p(r, \Theta)$.

What is $\vec{u} \cdot d\vec{S}$? In the case of a rigid piston moving at an amplitude U_0 which is in parallel to all surface vectors $d\vec{S}$ the product reduces to $U_0 dS$.

$$dp(r', t) = \frac{j\rho_0 ck}{2\pi r'}U_0 d\vec{S}e^{j(\omega t - kr')} \tag{15}$$

The entire pressure felt at $p(r, \Theta)$ is the integral over the whole piston:

$$p(r', t) = \frac{U_0 j\rho_0 ck}{2\pi}\int_{piston} \frac{e^{j(\omega t - kr')}}{r'}d\vec{S} \tag{16}$$

Substituting Eq. 14 into the equation above the integral can't be evaluated because of the complex denominator. However, if we are reasonably far away from the piston the difference between r and r' is geglectable. Thus, we only substitute r' in the numerator:

$$p(r', t) = \frac{U_0 j\rho_0 ck}{2\pi r}e^{j(\omega t - kr)}\int_{piston} e^{jk\sigma \sin\Theta \cos\psi}d\vec{S} \tag{17}$$

In the next step we have to specify what we mean with the surface element dS. For the integration we have to specify what dS means in the polar coordinates σ and ψ. In polar coordinates the surface element is $dS = \sigma d\sigma d\psi$. Note the additional σ:

$$p(r', t) = \frac{U_0 j\rho_0 ck}{2\pi r}e^{j(\omega t - kr)}\int_0^a \left[\int_0^{2\pi} e^{jk\sigma \sin\Theta \cos\psi}d\psi\right]\sigma d\sigma \tag{18}$$

The integral in brackets is a Bessel function:

$$J_m(x) = \frac{-j^n}{2\pi}\int_0^{2\pi} e^{jx\cos\psi}\cos(m\psi)d\psi \tag{19}$$

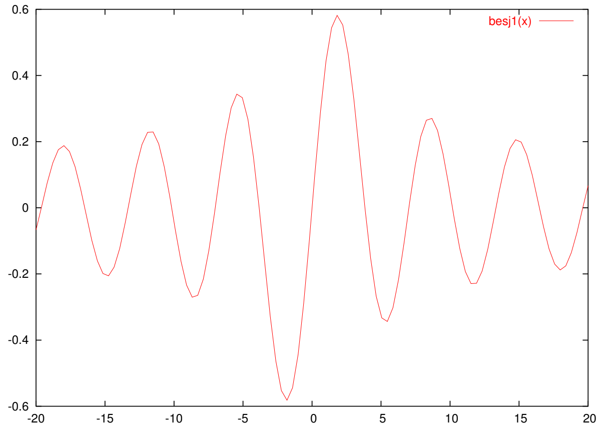

Figure 2: j1st Bessel function

we can then solve the inner integral of Eq. 18 by setting $x = k\sigma \sin\Theta$ and $m = 0$:

$$p(r', t) = \frac{U_0 j \rho_0 ck}{2\pi r} e^{j(\omega t - kr)} \int_0^a J_0(k\sigma \sin\Theta)\sigma d\sigma \tag{20}$$

The second integral can be solved by the following relation:

$$\int x J_0(x) dx = x J_1(x) \tag{21}$$

where $x = k\sigma \sin\Theta$. We have to be a bit cautious with the chain rule:

$$p(r', t) = \frac{U_0 j \rho_0 ck}{2\pi r} e^{j(\omega t - kr)} \frac{a J_1(ka \sin\Theta)}{k \sin\Theta} \tag{22}$$

The function J_1 is the j1st Bessel function which is shown in Fig. 2.

To compare the above equation with the simple source we multiply by a/a:

$$p(r', t) = \underbrace{\frac{U_0 j \rho_0 cka^2}{2\pi r} e^{j(\omega t - kr)}}_{\text{simple source}} \underbrace{\frac{J_1(ka \sin\Theta)}{ka \sin\Theta}}_{\text{directivity}} \tag{23}$$

Eq. 23 can be split up into the simple source and a so called directivity term: The sound is no longer radiated in all directions. Depending on the product ka it is more or less directional. The factor ka can be rewritten in the form:

$$ka = \frac{2a\pi}{\lambda} \tag{24}$$

where a is the radius of the piston and λ is the wavelength of the radiated sound signal. This means that Eq. 24 is the ratio between the size of the loudspeaker and the wavelength.

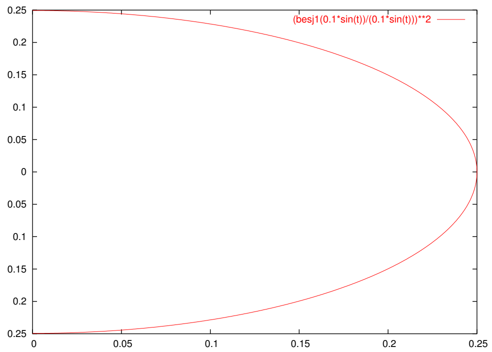

Figure 3: Directivity for $ka = 0.1$ and $ka = 10$

The directivity patterns look substantially different if the loudspeaker is smaller than the wavelength or larger. We have plotted the intensities (directivity squared) for the two cases $ka < 1$ and $ka > 1$ in Fig. 3.

References

Kinsler, L. E. (2000). *Fundamentals of acoustics.* Wiley, New York.