The wave equation of sound

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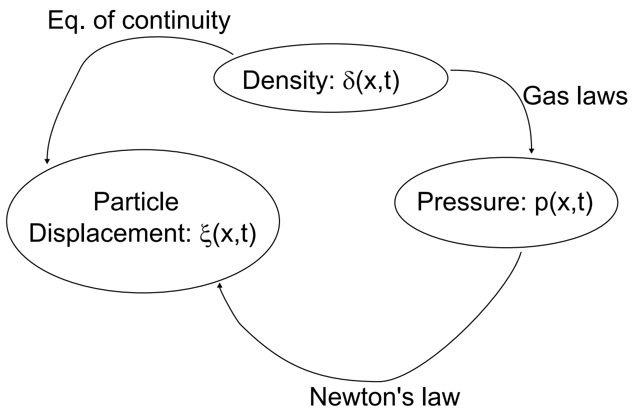

Figure 1: The relations between the different variables

1 Introduction

The derivation here is based on the book by Morse (1948). Fig. 1 shows the relations between the different variables. In particular we have to use the equation of continuity, Newton's law and the gas laws.

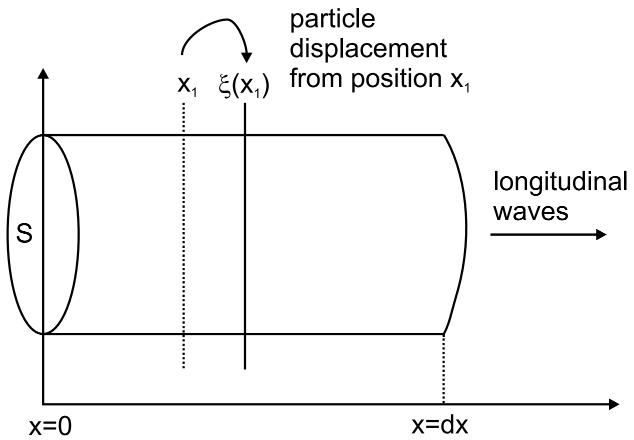

Figure 2: Geometry: a cylinder with length dx and surface S.

Fig. 2 shows the geometry. We look at a cylinder of air of length dx and with surface S. The length dx is so small that any variable changes linearly from the one side of the cylinder to the other one.

2 Definitions

- $\xi(x,t)$: Particle discplacement
- $\frac{\partial \xi(x,t)}{\partial t}$: Particle velocity
- ρ_0: Equilibrium density of the gas
- $\rho(x,t)$: Actual density
- $\delta(x,t)$: Relative change in density. $\rho(x,t) = \rho_0[1 + \delta(x,t)]$
- p_0: Equilibrium pressure
- $p(x,t)$: Change in pressure from the equilibrium pressure

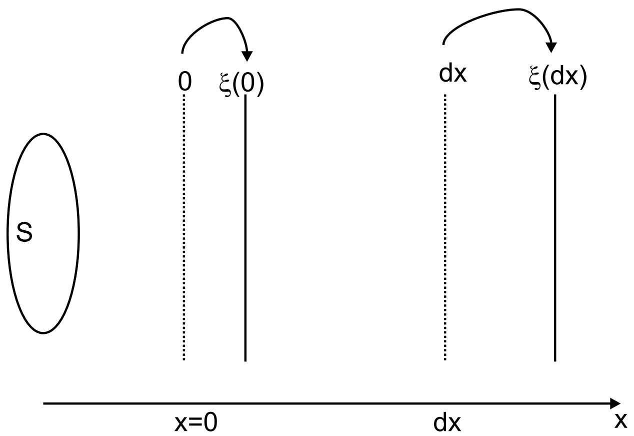

Figure 3: Particle displacements at two different positions: $x = 0$ and $x = dx$.

3 Equation of continuity

The equation of continuity establishes a relation between particle density $\delta(x,t)$ and displacement $\xi(x,t)$. Consider a scenario as shown in figure Fig. 3. We look at paricle displacements at the position 0 and dx. The *change* of the particle displacement between 0 and dx shall be linear, thus $\partial\xi/\partial x$ is constant.

The main assumption is here that the mass of the air between $x = 0$ and $x = dx$ must be conserved — with or without particle displacement:

$$\rho_0 S dx = \rho_0 V_0 = const \tag{1}$$

Thus, we are calaculating the mass of the air with and without displacement. We first calculate the volume and then the mass.

The volume of the air without discplacement is:

$$V_0 = S dx \tag{2}$$

The volume with displacements ξ is the volume without discplacement V_0 corrected by the displacements at $x = 0$ and $x = dx$. Since we assume linearity of the displacement we can approximate $\xi(dx)$ in a Taylor series and get:

$$\xi(dx) = \xi(0) + \frac{\partial\xi}{\partial x}dx \tag{3}$$

The volume with displacement is therefore:

$$V = S[dx + \xi(dx) - \xi(0)] = S dx + S\frac{\partial\xi}{\partial x}dx \tag{4}$$

Now we have the volume for the disturbed and the undisturbed case and we can set up a relation for the mass:

$$\rho(x,t)[S dx + S dx\frac{\partial\xi}{\partial x}] = \rho_0 S dx \tag{5}$$

With $\rho(x,t) = \rho_0[1 + \delta(x,t)]$ we get:

$$\rho_0 S dx[1 + \delta(x,t) + \frac{\partial\xi}{\partial x} + \underbrace{\delta(x,t)\frac{\partial\xi}{\partial x}}_{<<1}] = \rho_0 S dx \tag{6}$$

The last product on the left side can be omitted. So, we get:

$$\delta(x,t) + \frac{\partial\xi}{\partial x} = 0 \tag{7}$$

which is the equation of continuity.

4 Compressibility of the perfect gas

In this section we are deriving a relation between the density δ and the pressure p. We assume that no heat is generated during wave propagation: $dQ = 0$.

$$dQ = \frac{\partial Q}{\partial V} dV + \frac{\partial Q}{\partial P} dP \tag{8}$$

The partial derivatives can be expressed with the specific heats of the gas:

$$dQ = T \left(\frac{C_p}{V_0} dV + \frac{C_V}{P_0} dP \right) \tag{9}$$

and we get:

$$\frac{C_p}{V_0} dV = -\frac{C_V}{P_0} dP \tag{10}$$

The left hand side of Eq. 10 can be expressed in terms of the particle displacement if one considers the following relation:

$$V = \underbrace{Sdx}_{V_0} + \underbrace{Sdx \frac{\partial \xi}{\partial x}}_{dV} \tag{11}$$

On the right hand side of Eq. 10 dP corresponds directly to $p(x,t)$ and we get:

$$p = -\gamma_c P_0 \frac{\delta \xi}{\delta x} \qquad \text{with} \qquad \gamma_c = \frac{C_P}{C_V} \tag{12}$$

Now we take the equation of continuity (Eq. 3) and replace the partial derivative by the density:

$$p = \gamma_c P_0 \delta(x,t) \tag{13}$$

Finally we've got a relation between density and pressure.

5 Newton's law

To apply Newton's law we have to find an expression for the force acting on a small mass element (see Fig. 4) of the size Sdx.

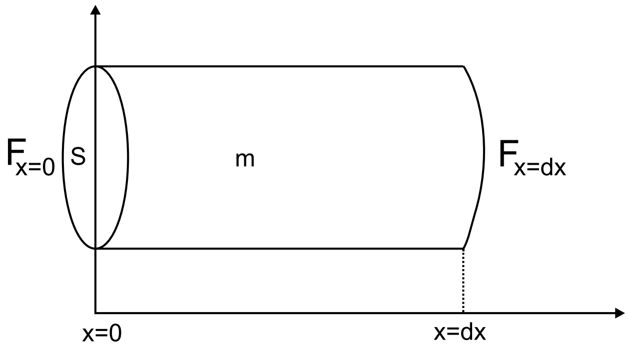

Figure 4: Forces at the two surfaces $x = 0$ and $x = dx$ accelerate the air in the cylinder.

The force acting upon the surface S at $x = 0$ is $F_{x=0} = [P_0 + p(x)]S$ and the force at the surface S at $x = dx$ is $F_{x=dx} = [P_0 + p(x) + \frac{\partial p}{\partial x} dx]S$. Thus, the net force is:

$$F = F_{x=dx} - F_{x=0} = \frac{\partial p}{\partial x} S dx \tag{14}$$

This force is accelerating the air in the cylinder which creates a counter-force which is

$$F = -ma = -\rho_0 S dx \frac{\partial^2 \xi}{\partial t^2} \tag{15}$$

Equating Eq. 15 with Eq. 14 gives:

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial^2 \xi}{\partial t^2} \tag{16}$$

which is called the linear Euler equation. It relates particle displacement to pressure.

6 Wave equation

We can now eliminate the pressure p in Eq. 12 with the help of Eq. 16. This gives us a wave equation for the particle displacement:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} \tag{17}$$

where

$$c = \sqrt{\frac{p_0 \gamma_c}{\rho_0}} \tag{18}$$

is the speed of sound.

To get a wave equation for the pressure we take the partial derivative $\frac{\partial}{\partial x}$ of Eq. 15 and change the order of the differentiation:

$$-\frac{\partial^2 p}{\partial x^2} = \rho_0 \frac{\partial^2}{\partial t^2} \underbrace{\frac{\partial \xi(x,t)}{\partial x}}_{-\delta(x,t)} \tag{19}$$

The density $\delta(x,t)$ can be eliminated by Eq. 13 and we finally get:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p(x,t)}{\partial t^2} \tag{20}$$

where c is again defined by Eq. 18.

References

Morse, P. M. (1948). *Vibration and Sound.* McGraw-Hill, New York.